Selecting Systems of Linear Equations

Systems of linear equations provide opportunities for students to think about lines in new ways, exploring their interactions in various representations and developing a greater conceptual understanding of linear equations themselves (National Council of Teachers of Mathematics 2000). Unfortunately, students often learn to solve systems of equations procedurally (Afamasaga-Fuata'l 2006). They are typically taught several methods for solving a system, such as substitution, combination (or elimination), or graphing, then given two linear equations to solve. The activity below provides students with an opportunity to approach systems of linear equations in a different way, allowing them to consider the structure of the equations before they begin solving a system. Students also have opportunities to make strategic mathematical decisions and discuss their reasoning, not just their solution process. It is meant to complement, rather than replace, typical problems involving systems of linear equations.

Background

The overarching goal of this activity is to engage students in the Common Core (2010) Standards for Mathematical Practice (SMPs). In particular, students define the problem they will solve (SMP1) and have opportunities to look at the mathematical structure of linear equations and to inform their mathematical decisions (SMP7). These practices, together with an awareness of what tools (e.g., graphing technology, algebraic techniques) might be helpful (SMP5), are important because they place more of the mathematical burden on students than when they simply execute a procedure. Furthermore, the activity provides formative assessment information about students' developing conceptual understanding rather than only information about their procedural success.

The relevant content standards from Common Core (2010) involve solving systems of equations in multiple ways (A-REI.6, A-REI.11). The activity also promotes consideration of the advantages and disadvantages of each method. The activity is designed to be used after basic lessons on solving systems of two linear equations (see Allen 2013 for an introductory lesson and Bush 2010 for a lesson comparing methods). It assumes students have some experience with solving systems by substitution and combination but are still developing their procedural and conceptual understanding of these topics. Ideally, students would also have graphing technology available and would be aware of the connections between systems of linear equations and intersections of graphed lines.

The Activity

Setting Up

The activity is built around a set of 8 linear equations. It begins by asking students to orient themselves to the equations, noting that they are linear but in different forms. This orientation is probably best achieved by the teacher leading a brief discussion about the equations. The students can recall the meaning of linearity and observe the range of equations so they can make informed choices throughout the rest of the activity.

Students can then be told that they will be choosing equations from the set to form systems that they will solve using three methods—substitution, combination, and graphing. Even though there are no “wrong” choices, they should be encouraged to make strategic choices and be ready to explain why they made those choices. They should also be encouraged to think across all three problems because their choices in one affect the others.

Working on Part 1

Students should have an opportunity to make their choices and work through Part 1, either individually or in small groups. Students who progress quickly can be encouraged to think about alternative choices (good or bad) or continue to Part 2.

Although this activity provides opportunities to practice solving systems of linear equations, the emphasis overall should be placed on the decisions students make along the way because these decision points can lead to conceptually-oriented discussions and productive arguments as students defend or amend their choices. As students work, the teacher can use the supplied monitoring sheet, making note of students' strategic choices (i.e., those leading to a straightforward application of a method) or their unexpected choices (i.e., those not particularly suited to the method) and then subsequently selecting and sequencing (Smith and Stein 2011) these choices in the mid-lesson discussion.

Mid-Lesson Discussion

When most of the students have completed Part 1 of the activity, a mid-lesson discussion can move in order through problems 1–3, with the attention directed away from answers and procedures. (Thankfully, the activity makes it difficult to talk about answers or specific steps because students likely chose different pairs of equations for each problem.) Instead, the teacher can frame the discussion around the students' decisions and their reasoning. Next, we provide potential teacher questions and mathematical ideas that can be discussed.

"What were you looking for when choosing equations for *substitution*?" Students can explain that they looked for equations already solved for a variable (D or F) or ones easily solved for a variable (B or E), which highlights a key facet of the substitution method. The teacher can then press further, asking how the students went about choosing their second equation once they had chosen a first. Here the teacher can rely on the monitoring sheet by calling on students who, perhaps, chose the same first equation but a different second equation. Another interesting idea to discuss is the notion that an equation does not need to be completely solved for a variable to be used in substitution. For example, equation B is $2x = 12$ and the term $2x$ appears in equations C and E. Students could choose to substitute 12 in place of $2x$ in those equations, allowing them to straightforwardly determine y. Similarly, E can be used to substitute directly into C. These techniques can broaden students' notions of substitution because they often think of it as a rigid procedure in which an equation must be completely solved for one variable.

"What were you looking for when choosing equations for *combination*?" Here the students probably did not select one equation at a time, as with substitution, but rather examined equations pairwise, looking for two equations that could be easily combined to eliminate a variable. Examples are B and C, A and G, or G and H, with the last pair actually combining to yield $0 = 0$, indicating identical lines. Some possible follow-up questions would be to ask whether students looked for equations that could be added (e.g., B and C) or whether they had to be subtracted or scaled first (e.g., G and H). It might also be worth bringing up that equations do not have to be in general form to be combined. For example, one might choose to subtract equation E from equation C to eliminate the y variable, even though equation E is not in general form. This highlights flexibility in the combination method rather than drilling a rigid procedure.

"What were you looking for when choosing equations for *graphing*?" It is likely that students chose equations in slope-intercept form (D or F) or equations that represent vertical or horizontal lines (B or D, respectively). One important point is the precision of the solution—that is, whether the intersection was clearly identifiable (and how the student knew) or whether it was an estimate. Returning to the issue of choice, the teacher might point out that a system with an intersection that can only be estimated graphically might be better suited for the substitution or combination methods, which give exact solutions. A powerful advantage of graphing, however, can be an immediate sense of parallel (C and E, addressed in problem 5) or coinciding lines (G and H) and the ability to view more than two lines simultaneously, immediately revealing the nature of their intersections (see Figure 1). Later, problem 7 addresses solving a system of three equations, which is most easily approached graphically (e.g., A, E, and F).

"What are the advantages and disadvantages of each solution method?" or "Knowing what you know now, would you change your choices?" These questions can wrap up the initial discussion and give the teacher an opportunity to emphasize that the various methods are all valid, but that certain equations might be easier with certain methods. It also allows the teacher to assess the sophistication with which students are thinking about linear equations and whether they have progressed in their conceptual understanding. Another potential question that our students have thoroughly enjoyed debating is: **Which equations would be the worst choice for each solution method?**

Working on Part 2

Students can then be asked to work through Part 2, with a discussion of this section being optional. These problems prompt students to return to the linear equations and think about them in slightly different ways. Problem 4 asks about the two unchosen equations. These may have been leftover for no particular reason, but we have found some of our students specifically avoiding certain equations, such as those involving fractions (F), those involving zero (G and H), or a certain form of equation they dislike. These aversions provided valuable information as we planned subsequent lessons.

Problem 5 draws special attention to equations C and E because these lines are parallel. Students may initially want to suggest a particular method to solve this system, but further thinking reveals that regardless of method, a solution cannot be reached. This problem helps illustrate that the absence of a solution is not due to an inadequacy in the solution method—it is a characteristic of the relationship between the two lines themselves.

Problem 6 allows for the possibility of solutions by inspection rather than by carrying out a formal procedure. The most obvious candidate for this approach is the system composed of equations B and D because they respectively yield the x-value and y-value of the solution. Students may also be able to immediately see the relationship between G and H.

Extensions

The two extension problems do not directly address the learning goals related to systems of two linear equations but they promote mathematical curiosity and exploration and thus relate to the SMPs. Problem 7 deals with the extension to a system of three equations. In most cases, there will not be a solution to such a system but one can arise if the three lines happen to intersect in a single point, as do A, E, and F. This is most easily seen graphically, and if one attempts to execute the combination method with three equations, they are quickly led astray because this method presupposes that a solution exists (otherwise you cannot combine the x's or y's as like-terms).

Problem 8 involves combinatorics—specifically, counting all the possible pairwise intersections of a set of 8 distinct lines, or more generally, n distinct lines. One way to think about this problem is that each line can intersect all the others—$n(n - 1)$ intersections—but this double counts each intersection point, so the general solution is $n(n - 1)/2$. Students may then consider ways in which the number of intersections can be reduced, which ties in with problem

7. They may also wish to see all the intersections within the activity's set of 8 lines, in which case a graphing tool is useful.

Conclusion

Most problems involving systems of linear equations begin by giving students two linear equations. Because the equations are fixed, it is natural for students' attention and energy to go toward the solution procedures. Our activity complements those problems by fixing the solution procedures and allowing choice with the equations, thus forcing students' attention toward the equations' structure. Moreover, students can interact with one another by discussing and justifying each of the decision points in the activity. This supports the SMPs in several ways. First, students define the problem with their choices (SMP1). Second, they have opportunities to argue for their choices or critique others' (SMP3). Third, their decision-making can be based on their perception of the structure of the linear equations (SMP7). Finally, they have the opportunity to use tools in strategic ways (SMP5) and to consider the precision of their work with those tools (SMP6), such as the fact that graphs can be less precise than algebraic solutions. Overall, this activity can create new, complementary experiences for students as they work with systems of linear equations.

References

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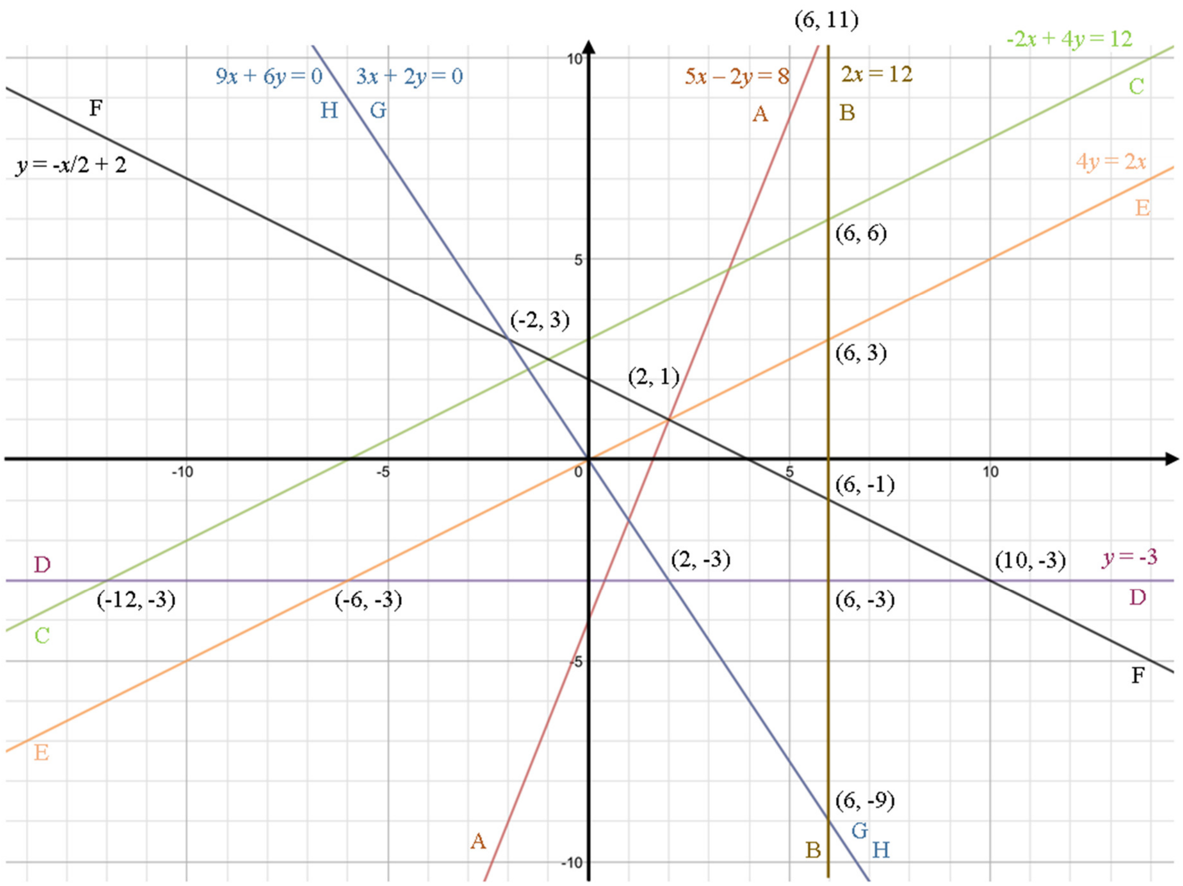

Figure 1. A graph of eight lines with various intersection points.